Exercise 1: Show that the second Piola-Kirchhoff stress tensor satisfied the following relations

(a) $\boldsymbol{S} = \boldsymbol{S}^T$ (Symmetry) and (b) $\boldsymbol{S} = \boldsymbol{S}^*$ (Objectivity)

Exercise 2: With respect to the principal axes, the invariants of the Cauchy-Green tensor $\boldsymbol{C}$ are,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \; . \qquad \text{(a)}$$

Here $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the principal stretches. For an isotropic, incompressible material show that,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2}, \quad I_2 = \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}.$$

Note: Stretch ratio λ, is defined as the ratio of the current length to the initial length of a linear segment after deformation (see kinematics presentation). If we consider a cube of material, in three dimensions, with sides A, B, C at the initial configuration, these sides will become a, b, c after deformation. Thus, a/A $= \lambda_1$, b/B $= \lambda_2$, c/C $= \lambda_3$ are the three stretch ratios.

Exercise 3: Use the expressions for the invariants in exercise (1) to show that,

$$\frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I}, \quad \frac{\partial I_2}{\partial \boldsymbol{C}} = I_1\boldsymbol{I} - \boldsymbol{C}, \quad \frac{\partial \lambda_i^2}{\partial \boldsymbol{C}} = \boldsymbol{A}_i \otimes \boldsymbol{A}_i$$

where $\lambda_i^2 (i = 1,2,3)$ are the principal stretches and $\boldsymbol{A}_i \;\; (i = 1,2,3,)$ the principal directions of the deformation tensor $\boldsymbol{C}$.

Hint: use the following relation for the derivative of scalar function of a tensor variable (B1.144)

$$\frac{\partial \mathcal{W}}{\partial \boldsymbol{T}} = \sum_{i=1}^{3} \frac{\partial \mathcal{W}}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i \quad \text{where} \qquad \mathcal{W}(\boldsymbol{T}) = \phi(\lambda_1, \lambda_2, \lambda_3) \quad .$$

Here $\lambda_i \;\; (i = 1,2,3,)$ are the principal values of $\boldsymbol{T}$, with the coresponding directions $\boldsymbol{n}_i$.

Exercise 4: Show that, for an isotropic material, the tensors $\boldsymbol{U}$, $\boldsymbol{C}$ and $\boldsymbol{S}$ have the same principal directions. Recall that

1: $(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{v} \cdot \boldsymbol{a})(\boldsymbol{u} \otimes \boldsymbol{b}) = (\boldsymbol{u} \otimes \boldsymbol{b})(\boldsymbol{v} \cdot \boldsymbol{a})$

2: $(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)(\boldsymbol{n}_j \otimes \boldsymbol{n}_j) = \begin{cases} 0 \;\; if \;\; i \neq j \\ (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \;\; if \;\; i = j \end{cases}.$

Exercise 5: Consider the general expression for the energy function,

$$\Phi(I_1, I_2, I_3) = \sum_{i,\,j,\,k=0}^{\infty} C_{ijk}(I_1 - 3)^i (I_2 - 3)^j (I_3 - 1)^k$$

where the invariants are expressed in terms of the principal stretches,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 .$$

Show that: (1) in the reference configuration ($\boldsymbol{S} = \boldsymbol{0}$ and $\boldsymbol{C} = \boldsymbol{I}$) we have $C_{000} = 0$. (2) in the reference configuration, the coefficients C_{100}, C_{010}, C_{001} satisfy the following relation $C_{100} + 2C_{010} + C_{001} = 0$.